Unit 9: Vectors and Complex Numbers

Precalculus Honors | Standards PC.NR.2 & PC.NR.3

Brian Risden

Unit Overview

We will explore the geometry and algebra of vectors and complex numbers, focusing on their representations and applications.

- **Vectors (PC.NR.2)** — Representing directed line segments, performing operations (component-wise and graphically), and modeling force and velocity.
- **Complex Numbers (PC.NR.3)** — Representing numbers on the complex plane, converting between rectangular and polar forms, and interpreting operations geometrically.

Vectors: Representation & Modeling

Vector Basics

A **vector** represents a quantity with both **magnitude** (length) and **direction**.

Form	Notation	Formula
Component Form	$\mathbf{v} = \langle x, y \rangle$	$x = v_2 - v_1, y = v_2 - v_1$
Magnitude	$\|\mathbf{v}\|$	$\sqrt{x^2 + y^2}$
Direction Angle	θ	$\tan\theta = \frac{y}{x}$

Interactive: Vector Component Explorer

```
//| panel: sidebar
viewof v_x = Inputs.range([-10, 10], {value: 4, step: 1, la
viewof v_y = Inputs.range([-10, 10], {value: 3, step: 1, la
```

```
//| panel: fill
v_mag = Math.sqrt(v_x**2 + v_y**2)
v_ang = Math.atan2(v_y, v_x) * 180 / Math.PI

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-11, 11], grid: true},
  y: {domain: [-11, 11], grid: true},
  marks: [
    Plot.arrow([[0, 0], [v_x, v_y]], {stroke: "#332288", st
    Plot.text([[v_x, v_y + 1]], {text: [`v = ${v_x}, ${v_y
    Plot.text([[-9, 9]], {
      text: [`Magnitude: ${v_mag.toFixed(2)} | Angle: ${v_a
      fontSize: 22, fontWeight: "bold", fill: "#117733"
    })
```

Vector Addition & Resultants

To find the **resultant vector** $\mathbf{u} + \mathbf{v}$:

1. **Algebraically:** Add corresponding components: $\langle u_1 + v_1, u_2 + v_2 \rangle$.
2. **Geometrically:** Use the **Tail-to-Head** method or the **Parallelogram Law**.

Scalar Multiplication: Multiplying a vector by k changes its magnitude (and direction if $k < 0$) but stays on the same line.

Interactive: Addition & Parallelogram Law

```
//| panel: sidebar
viewof u_x = Inputs.range([-5, 5], {value: 3, step: 1, labe
viewof u_y = Inputs.range([-5, 5], {value: 1, step: 1, labe
viewof w_x = Inputs.range([-5, 5], {value: 1, step: 1, labe
viewof w_y = Inputs.range([-5, 5], {value: 4, step: 1, labe
```

```
//| panel: fill
r_x = u_x + w_x
r_y = u_y + w_y

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-1, 10], grid: true},
  y: {domain: [-1, 10], grid: true},
  marks: [
    Plot.arrow([[0, 0], [u_x, u_y]], {stroke: "#117733", st
    Plot.arrow([[0, 0], [w_x, w_y]], {stroke: "#44AA99", st
    // Parallelogram dashed lines
    Plot.line([[u_x, u_y], [r_x, r_y]], {stroke: "#aaa", st
```

Practice: PC.NR.2

1. Find the component form of a vector with magnitude 10 and direction angle 150°.

$x = 10\cos(150^\circ) = -5\sqrt{3}, \quad y = 10\sin(150^\circ) = 5 \to \langle -5\sqrt{3}, 5\rangle$

2. Given $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle 5, 4\rangle$, find $2\mathbf{u} - \mathbf{v}$.

$\langle 4, -6\rangle - \langle 5, 4\rangle = \langle 4-5, -6-4\rangle \to \langle -1, -10\rangle$

3. An airplane flies at 400 mph at $N30^\circ E$. A wind blows at 50 mph from the West. Find the resultant velocity.

$\mathbf{p} = \langle 400\sin 30, 400\cos 30\rangle = \langle 200, 346.4\rangle$
$\mathbf{w} = \langle 50, 0\rangle \to \mathbf{r} = \langle 250, 346.4\rangle$ Mag ≈ 427.2 mph

Complex Numbers in the Complex Plane

Rectangular vs. Polar Form

The complex number $z = a + bi$ can be mapped to (a, b) on the complex plane.

Form	Representation	Conversion
Rectangular	$z = a + bi$	$a = r\cos\theta, b = r\sin\theta$
Polar	$z = r(\cos\theta + i\sin\theta)$	$r = \sqrt{a^2 + b^2}, \tan\theta = \frac{b}{a}$

Insight: These represent the same number because they describe the same point (a, b) using either Cartesian coordinates or distance/angle from the origin.

Interactive: Complex Form Converter

```
//| panel: sidebar
viewof c_a = Inputs.range([-10, 10], {value: 3, step: 1, la
viewof c_b = Inputs.range([-10, 10], {value: 3, step: 1, la
```

```
//| panel: fill
c_r = Math.sqrt(c_a**2 + c_b**2)
c_theta = Math.atan2(c_b, c_a) * 180 / Math.PI

html`<div style="font-size: 26px; padding: 20px; background
  <p><span style="color:#332288"><b>Rectangular:</b></span
  <p><span style="color:#117733"><b>Polar:</b></span> ${c_r
</div>`

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-11, 11], grid: true, label: "Real"},
  y: {domain: [-11, 11], grid: true, label: "Imaginary"},
  marks: [
    Plot.dot([[c_a, c_b]], {r: 6, fill: "#AA4499"}),
```

Operations in Polar Form

Polar form makes multiplication and division much simpler:

Multiplication: Multiply r's, Add θ's
$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Division: Divide r's, Subtract θ's
$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Geometric Interpretations (PC.NR.3.2)

- **Addition:** Follows the parallelogram rule (identical to vectors).
- **Conjugation:** A reflection over the Real (x) axis.
- **Multiplication by i:** A 90° counter-clockwise rotation.
- **Multiplication by $r(\cos\theta + i\sin\theta)$:** A scaling by r and rotation by θ.

Practice: PC.NR.3

1. Write $z = -2 + 2i\sqrt{3}$ in polar form.

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$
$\theta = \mathsf{atan2}(2\sqrt{3}, -2) = 120^\circ$ $z = 4(\cos 120^\circ + i \sin 120^\circ)$

2. Find $z_1 z_2$ for $z_1 = 3(\cos 40^\circ + i \sin 40^\circ)$ and $z_2 = 2(\cos 20^\circ + i \sin 20^\circ)$.

$r = 3 \cdot 2 = 6, \quad \theta = 40^\circ + 20^\circ = 60^\circ$
$6(\cos 60^\circ + i \sin 60^\circ) = 3 + 3i\sqrt{3}$

3. Geometrically, what happens to z when you find its conjugate $\bar{z}$?

The imaginary part changes sign: $(a, b) \to (a, -b)$. This results in a **reflection across the horizontal (real) axis.**

Summary

Operation	Algebra	Geometry
Vector Addition	$\langle x_1 + x_2, y_1 + y_2 \rangle$	Parallelogram/Head-to-Tail
Complex Addition	$(a + c) + (b + d)i$	Parallelogram
Complex Mult.	$r_1 r_2, \theta_1 + \theta_2$	Rotation + Scaling
Conjugation	$a - bi$	Reflection over Real Axis

$z = r(\cos\theta + i\sin\theta) \quad \mathbf{v} = \langle \|\mathbf{v}\|\cos\theta, \|\mathbf{v}\|\sin\theta \rangle$ "‘