Trigonometry & Periodic Phenomena

Precalculus Honors

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Area of a Triangle

When we know the lengths of all three sides (a, b, c), we can determine the area without needing the height by utilizing Heron's Formula.

- **Semi-perimeter (s):** $s = \frac{a+b+c}{2}$
- **Heron's Formula:** Area $= \sqrt{s(s-a)(s-b)(s-c)}$

Sample Problem: A surveyor is measuring a triangular plot of land. The side lengths are 40 meters, 50 meters, and 70 meters. What is the area of the plot? . . . **Solution:** $s = \frac{40+50+70}{2} = 80$.
Area $= \sqrt{80(40)(30)(10)} = \sqrt{960000} \approx 979.8$ m^2

Laws of Sines and Cosines

These laws allow us to find unknown measurements in non-right (oblique) triangles, which is vital for analyzing resultant forces and navigation.

- **Law of Sines:** $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- **Law of Cosines:** $c^2 = a^2 + b^2 - 2ab\cos C$

Sample Problem: Two tugboats are pulling a barge. The cables make an angle of 45° with each other. If the lengths of the cables are 100 ft and 120 ft, how far apart are the tugboats? . . .
Solution: Use Law of Cosines.
$x^2 = 100^2 + 120^2 - 2(100)(120)\cos(45^\circ)$ $x \approx 86.1$ ft

Interactive Triangle Solver

Manipulate the three side lengths. Notice what happens when the sides violate the Triangle Inequality Theorem. If valid, Heron's Formula calculates the area.

```
//| panel: sidebar
viewof side_a = Inputs.range([1, 10], {value: 6, step: 0.5,
viewof side_b = Inputs.range([1, 10], {value: 8, step: 0.5,
viewof side_c = Inputs.range([1, 10], {value: 10, step: 0.5
```

```
//| panel: fill
// Math Logic
isValid = (side_a + side_b > side_c) && (side_a + side_c >
s_val = (side_a + side_b + side_c) / 2
area = Math.sqrt(s_val * (s_val - side_a) * (s_val - side_b
// Use Law of Cosines to find angle A for drawing purposes
angle_A = Math.acos((side_b**2 + side_c**2 - side_a**2) / (

Plot.plot({
  aspectRatio: 1,
```

Restricting Domains for Inverses

To define an inverse function, the original function must pass the horizontal line test. We restrict the domain of $y = \sin(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

```
//| panel: sidebar
viewof show_inverse = Inputs.toggle({label: "Show Inverse 
```

```
//| panel: fill
Plot.plot({
  aspectRatio: 1,
  x: {domain: [-Math.PI, Math.PI], label: "x"},
  y: {domain: [-Math.PI, Math.PI], label: "y"},
  grid: true,
  marks: [
    Plot.ruleX([0]), Plot.ruleY([0]),
    // Line of symmetry y=x
    Plot.line([[-Math.PI, -Math.PI], [Math.PI, Math.PI]], 

    // Full Sine Wave (faded)
```

Arc Length and Sector Area

In calculus, working with circular motion requires converting our degree measures into radians.

For a circle of radius r and central angle θ (in radians): * **Arc Length (s):** $s = r\theta$ * **Sector Area (A):** $A = \frac{1}{2}r^2\theta$

Sample Problem: A sprinkler sprays water a distance of 8 feet and rotates through an angle of $\frac{2\pi}{3}$ radians. Find the area of the lawn watered. . . . **Solution:**
$A = \frac{1}{2}(8)^2(\frac{2\pi}{3}) = 32(\frac{2\pi}{3}) = \frac{64\pi}{3} \approx 67.02$ ft^2

Interactive Arc Length and Sector Area

Adjust the radius r and the central angle θ (in radians) to see how the arc length and sector area change proportionally.

```
//| panel: sidebar
viewof r_sec = Inputs.range([1, 10], {value: 5, step: 0.5,
viewof theta_sec = Inputs.range([0, 2*Math.PI], {value: 1.5
```

```
//| panel: fill
Plot.plot({
  aspectRatio: 1,
  x: {domain: [-11, 11]},
  y: {domain: [-11, 11]},
  grid: true,
  marks: [
    Plot.ruleX([0]), Plot.ruleY([0]),
    // Draw the faint full circle
    Plot.line(d3.range(0, 2*Math.PI + 0.1, 0.1), {
      x: d => r_sec * Math.cos(d),
      y: d => r_sec * Math.sin(d),
```

The Six Trigonometric Ratios

For any angle θ in standard position, let (x, y) be a point on its terminal side, and let $r = \sqrt{x^2 + y^2}$ be the distance from the origin.

- $\sin\theta = \frac{y}{r}$
- $\cos\theta = \frac{x}{r}$
- $\tan\theta = \frac{y}{x}$ $(x \neq 0)$
- $\csc\theta = \frac{r}{y}$ $(y \neq 0)$
- $\sec\theta = \frac{r}{x}$ $(x \neq 0)$
- $\cot\theta = \frac{x}{y}$ $(y \neq 0)$

Sample Problem: The point $(-3, 4)$ is on the terminal side of an angle θ. Find $\sec\theta$. . . . **Solution:** $r = \sqrt{(-3)^2 + 4^2} = 5$. Therefore, $\sec\theta = \frac{r}{x} = -\frac{5}{3}$.

Reference Angles & Exact Values

We can evaluate trig functions for any angle by finding its reference angle in the first quadrant, utilizing the properties $\pi - x$, $\pi + x$, and $2\pi - x$.

Sample Problem: Determine the exact value of $\cos(\frac{5\pi}{6})$ without a calculator.

Reference Angles & Exact Values

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Sample Problem: Determine the exact value of $\cos(\frac{5\pi}{6})$ without a calculator.

Solution: 1. The angle $\frac{5\pi}{6}$ is in Quadrant II. 2. The reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. 3. In Quadrant II, cosine is negative. 4. $\cos(\frac{5\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$

Inverse Trigonometric Functions

To create an inverse function, the original function must pass the horizontal line test. We must restrict the domains of our trigonometric functions to make them one-to-one.

- $y = \arcsin(x)$ outputs angles in $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $y = \arccos(x)$ outputs angles in $[0, \pi]$
- $y = \arctan(x)$ outputs angles in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Sample Problem: Evaluate $\arcsin(-\frac{1}{2})$. . . . **Solution:** We need an angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is $-\frac{1}{2}$. The answer is $-\frac{\pi}{6}$.

Solving Trigonometric Equations

When modeling real-life phenomena, we use inverse functions to solve equations, paying close attention to interval restrictions.

Sample Problem: Solve $2\sin(x) - \sqrt{3} = 0$ on the interval $[0, 2\pi)$.

. . . **Solution:** 1. Isolate the trig function: $\sin(x) = \frac{\sqrt{3}}{2}$ 2. Identify angles in the unit circle where the y-coordinate is $\frac{\sqrt{3}}{2}$. 3. $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$

Solving Triangles

We can find unknown measurements in right and non-right triangles using specific formulas and laws.

- **Area of a Triangle:** We can determine the area to solve problems, utilizing Heron's Formula when given the lengths of all three sides.
- **Laws of Sines and Cosines:** We will prove and apply both the Law of Sines and the Law of Cosines to find unknown measurements.
- **Applications:** These tools allow us to investigate real-world scenarios like surveying problems and resultant forces.

Radians and Circular Motion

Angles can be measured in degrees, but calculus requires radian measure.

- **Arc Length and Sector Area:** We will derive and apply the formulas for the length of an arc and the area of a sector in a circle.
- **Radian Conversions:** We must efficiently convert between degree and radian measures.
- **Quadrantal Angles:** We will develop the radian measure of the quadrantal angles and work with measures that are both in terms of π and those not in terms of π.

The Unit Circle Basics

By setting a circle's radius to 1, we unlock the core trigonometric values.

- **Special Right Triangles:** We will geometrically determine the values of sine, cosine, and tangent for $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.
- **Reference Angles:** The unit circle allows us to express trig values for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x.
- **The Six Ratios:** We will define the six trigonometric ratios in terms of x, y, and r using the unit circle centered at the origin.

Interactive Unit Circle

Explore how the coordinates change as the angle θ (in radians) increases both counterclockwise and clockwise. Notice the parametric interpretation where the coordinates are $(\cos(t), \sin(t))$.

```
//| panel: sidebar
viewof t = Inputs.range([0, 2*Math.PI], {value: 0.785, step
```

```
//| panel: fill
Plot.plot({
  aspectRatio: 1,
  x: {domain: [-1.5, 1.5], label: "Cosine (x)"},
  y: {domain: [-1.5, 1.5], label: "Sine (y)"},
  marks: [
    Plot.ruleX([0]), Plot.ruleY([0]),
    Plot.line({length: 100}, {x: (d, i) => Math.cos(i/100*2
    Plot.line([[0,0], [Math.cos(t), Math.sin(t)]], {stroke:
    Plot.dot([[Math.cos(t), Math.sin(t)]], {fill: "red", r:
    Plot.text([[Math.cos(t), Math.sin(t)]], {
      text: [ `(${Math.cos(t).toFixed(2)}, ${Math.sin(t).to
      dy: -15, fontWeight: "bold"
    })
  ]
})
```

Symmetry and Periodicity

The unit circle helps us understand the fundamental properties of trigonometric functions.

- **Symmetry:** We will explain symmetry, identifying which functions are odd and which are even.
- **Periodicity:** We will investigate the periodic nature of these functions by linking unit circle rotations to graphical representations.

Graphing Trigonometric Functions

We will graph these functions and analyze their structures to model real-life phenomena.

- **Key Features:** We must describe period, midline, amplitude, phase shift, intercepts, asymptotes, symmetries, domain, and range.
- **Advanced Analysis:** We will identify relative extrema and intervals where the function is increasing, decreasing, positive, or negative, using proper interval notation.

Interactive Transformations

Observe how modifying the parameters a, b, h, and k affects the amplitude, period, phase shift, and midline of $y = a\sin(b(x-h)) + k$.

```
//| panel: sidebar
viewof a = Inputs.range([-4, 4], {value: 1, step: 0.5, labe
viewof b = Inputs.range([0.5, 4], {value: 1, step: 0.5, lab
viewof h = Inputs.range([-3.14, 3.14], {value: 0, step: 0.2
viewof k = Inputs.range([-4, 4], {value: 0, step: 0.5, labe
```

```
//| panel: fill
//|
// d3 = require("d3@7")
// import {range} from "d3"

Plot.plot({
  x: {domain: [-2*Math.PI, 2*Math.PI], label: "x (radians)"
  y: {domain: [-6, 6], label: "y"},
  marks: [
```