Trigonometric Identities and Formulas

Precalculus Honors | Standards PC.PAFR.6.1 & 6.2

Brian Risden

Standards Overview

This presentation covers the manipulation and application of trigonometric identities and formulas as outlined in the South Carolina College- and Career-Ready Standards:

- **PC.PAFR.6.1:** Apply fundamental trigonometric identities (quotient, reciprocal, Pythagorean, even/odd, and cofunction) to simplify and verify.
- **PC.PAFR.6.2:** Apply sum, difference, double-angle, and half-angle formulas for sine, cosine, and tangent to solve problems.

PC.PAFR.6.1: Fundamental Identities

Reciprocal & Quotient Identities

Reciprocal: $\csc x = \frac{1}{\sin x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \cot x = \frac{1}{\tan x}$

Quotient: $\tan x = \frac{\sin x}{\cos x}$ $\quad \cot x = \frac{\cos x}{\sin x}$

Practice: Reciprocal Identities

1. Simplify: $\sin x \cdot \csc x$

$\sin x \cdot \left(\frac{1}{\sin x}\right) \to 1$

2. Simplify: $\frac{\sec \theta}{\csc \theta}$

$\frac{1/\cos \theta}{1/\sin \theta} = \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{1} \to \tan \theta$

3. Simplify: $\cos \beta \cdot \sec \beta - \sin^2 \beta$

$1 - \sin^2 \beta \to \cos^2 \beta$

Practice: Quotient Identities

1. Simplify: $\cos x \cdot \tan x$

$\cos x \cdot \left(\frac{\sin x}{\cos x}\right) \to \sin x$

2. Simplify: $\frac{\sin\alpha}{\cos\alpha \cdot \tan\alpha}$

$\frac{\sin\alpha}{\cos\alpha \cdot (\sin\alpha / \cos\alpha)} = \frac{\sin\alpha}{\sin\alpha} \to 1$

3. Simplify: $\cot\theta \cdot \sin\theta$

$\left(\frac{\cos\theta}{\sin\theta}\right) \cdot \sin\theta \to \cos\theta$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Practice: Pythagorean Identities

1. Simplify: $(1 - \sin^2 x)(\sec^2 x)$

$\cos^2 x \cdot \frac{1}{\cos^2 x} \rightarrow$ 1

2. Simplify: $\tan^2 \theta - \sec^2 \theta$

$\tan^2 \theta - (1 + \tan^2 \theta) \rightarrow$ -1

3. Verify: $\sin^2 \alpha \cdot \cot^2 \alpha + \sin^2 \alpha = 1$

$\sin^2 \alpha(\cot^2 \alpha + 1) = \sin^2 \alpha \cdot \csc^2 \alpha \rightarrow$ 1 = 1

Even/Odd & Cofunction Identities

Even/Odd:

$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \tan(-x) = -\tan x$

Cofunction:

$\sin(\frac{\pi}{2} - x) = \cos x, \quad \tan(\frac{\pi}{2} - x) = \cot x, \quad \sec(\frac{\pi}{2} - x) = \csc x$

Practice: Even/Odd Identities

1. **Simplify:** $\sin(-x) \cdot \csc x$

$-\sin x \cdot \frac{1}{\sin x} \rightarrow$ -1

2. **Simplify:** $\cos(-x) + \sin(-x)\tan x$

$\cos x - \sin x(\frac{\sin x}{\cos x}) = \frac{\cos^2 x - \sin^2 x}{\cos x} \rightarrow \frac{\cos 2x}{\cos x}$

3. **Simplify:** $\frac{\tan(-x)}{\sin(-x)}$

$\frac{-\tan x}{-\sin x} = \frac{\sin x / \cos x}{\sin x} \rightarrow \sec x$

Practice: Cofunction Identities

1. Simplify: $\cos(\frac{\pi}{2} - x) \cdot \csc x$

$\sin x \cdot \frac{1}{\sin x} \to 1$

2. Simplify: $\frac{\sin(\frac{\pi}{2} - x)}{\cos(\frac{\pi}{2} - x)}$

$\frac{\cos x}{\sin x} \to \cot x$

3. Simplify: $\tan(\frac{\pi}{2} - \theta) \cdot \tan\theta$

$\cot\theta \cdot \tan\theta = \frac{1}{\tan\theta} \cdot \tan\theta \to 1$

PC.PAFR.6.2: Compound Angle Formulas

Sum and Difference Formulas

Function	Sum Formula
Sine	$\sin(u+v) = \sin u \cos v + \cos u \sin v$
Cosine	$\cos(u+v) = \cos u \cos v - \sin u \sin v$
Tangent	$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$

Function	Difference Formula
Sine	$\sin(u-v) = \sin u \cos v - \cos u \sin v$
Cosine	$\cos(u-v) = \cos u \cos v + \sin u \sin v$
Tangent	$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

Practice: Sine Sum & Difference

1. **Exact value:** $\sin(75^\circ)$ **using** $45^\circ + 30^\circ$

$\sin 45 \cos 30 + \cos 45 \sin 30 = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} \to \frac{\sqrt{6}+\sqrt{2}}{4}$

2. **Exact value:** $\sin(15^\circ)$ **using** $45^\circ - 30^\circ$

$\sin 45 \cos 30 - \cos 45 \sin 30 = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \to \frac{\sqrt{6}-\sqrt{2}}{4}$

3. **Simplify:** $\sin(x + \pi)$

$\sin x \cos \pi + \cos x \sin \pi = \sin x(-1) + \cos x(0) \to -\sin x$

Practice: Cosine Sum & Difference

1. **Exact value:** $\cos(105^\circ)$ **using** $60^\circ + 45^\circ$

$\cos 60 \cos 45 - \sin 60 \sin 45 = \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \rightarrow \frac{\sqrt{2}-\sqrt{6}}{4}$

2. **Exact value:** $\cos(\frac{\pi}{12})$ **using** $\frac{\pi}{3} - \frac{\pi}{4}$

$\cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \rightarrow \frac{\sqrt{2}+\sqrt{6}}{4}$

3. **Simplify:** $\cos(x - \frac{\pi}{2})$

$\cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2} = \cos x(0) + \sin x(1) \rightarrow \sin x$

Practice: Tangent Sum & Difference

1. Exact value: $\tan(75^\circ)$ **using** $45^\circ + 30^\circ$

$\frac{\tan 45+\tan 30}{1-\tan 45 \tan 30} = \frac{1+\sqrt{3}/3}{1-(1)(\sqrt{3}/3)} \to 2+\sqrt{3}$

2. Exact value: $\tan(15^\circ)$ **using** $45^\circ - 30^\circ$

$\frac{\tan 45-\tan 30}{1+\tan 45 \tan 30} = \frac{1-\sqrt{3}/3}{1+(1)(\sqrt{3}/3)} \to 2-\sqrt{3}$

3. Find $\tan(u+v)$ **if** $\tan u = 3$ **and** $\tan v = 2$

$\frac{3+2}{1-(3)(2)} = \frac{5}{-5} \to$ -1

Double-Angle Formulas

Function	Double-Angle Formulas
Sine	$\sin 2u = 2\sin u\cos u$
Cosine	$\cos 2u = \cos^2 u - \sin^2 u$
Cosine	$\cos 2u = 2\cos^2 u - 1$
Cosine	$\cos 2u = 1 - 2\sin^2 u$
Tangent	$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$

Practice: Double-Angle Sine

1. If $\sin u = \frac{3}{5}$ **and** u **is in Q1, find** $\sin 2u$

$\cos u = \frac{4}{5} \to 2(\frac{3}{5})(\frac{4}{5}) \to \frac{24}{25}$

2. Simplify: $4 \sin x \cos x$

$2(2 \sin x \cos x) \to 2 \sin 2x$

3. Solve on $[0, 2\pi)$**:** $\sin 2x - \cos x = 0$

$2 \sin x \cos x - \cos x = 0 \to \cos x(2 \sin x - 1) = 0 \to$
$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Practice: Double-Angle Cosine

1. If $\cos u = -\frac{2}{3}$**, find** $\cos 2u$

$2(-\frac{2}{3})^2 - 1 = 2(\frac{4}{9}) - 1 = \frac{8}{9} - \frac{9}{9} \rightarrow -\frac{1}{9}$

2. Simplify: $1 - 2\sin^2(15^\circ)$

$\cos(2 \cdot 15^\circ) = \cos 30^\circ \rightarrow \frac{\sqrt{3}}{2}$

3. Verify: $\frac{1+\cos 2x}{2} = \cos^2 x$

$\frac{1+(2\cos^2 x - 1)}{2} = \frac{2\cos^2 x}{2} \rightarrow \cos^2 x = \cos^2 x$

Practice: Double-Angle Tangent

1. If $\tan u = \frac{1}{2}$, find $\tan 2u$

$\frac{2(1/2)}{1-(1/2)^2} = \frac{1}{1-1/4} = \frac{1}{3/4} \to \frac{4}{3}$

2. Simplify: $\frac{2\tan(22.5^\circ)}{1-\tan^2(22.5^\circ)}$

$\tan(2 \cdot 22.5^\circ) = \tan 45^\circ \to 1$

3. Find $\tan 2x$ if $\sin x = \frac{5}{13}$ in Q2

$\cos x = -\frac{12}{13}, \tan x = -\frac{5}{12} \to \frac{2(-5/12)}{1-(-5/12)^2} \to -\frac{120}{119}$

Half-Angle Formulas

Sine: $\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$

Cosine: $\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$

Tangent: $\tan \frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$

(Sign depends on the quadrant of $u/2$)

Practice: Half-Angle Sine

1. Exact value: $\sin(22.5^\circ)$

$\sqrt{\frac{1-\cos 45}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} \rightarrow \frac{\sqrt{2-\sqrt{2}}}{2}$

2. If $\cos u = \frac{1}{4}$ **and** u **is in Q4, find** $\sin \frac{u}{2}$

$u/2$ is in Q2, so sine is positive: $\sqrt{\frac{1-1/4}{2}} = \sqrt{\frac{3/4}{2}} \rightarrow \frac{\sqrt{6}}{4}$

3. Simplify: $\sqrt{\frac{1-\cos 80^\circ}{2}}$

$\sin(\frac{80^\circ}{2}) \rightarrow \sin 40^\circ$

Practice: Half-Angle Cosine

1. Exact value: $\cos(75^\circ)$ **using half of** 150°

$\sqrt{\frac{1+\cos 150}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} \rightarrow \frac{\sqrt{2-\sqrt{3}}}{2}$

2. If $\cos u = -\frac{1}{2}$ **and** u **is in Q3, find** $\cos \frac{u}{2}$

$u/2$ is in Q2, so cosine is negative: $-\sqrt{\frac{1+(-1/2)}{2}} = -\sqrt{1/4} \rightarrow -\frac{1}{2}$

3. Simplify: $2\cos^2(\frac{x}{2}) - 1$

$\cos(2 \cdot \frac{x}{2}) \rightarrow \cos x$

Practice: Half-Angle Tangent

1. Exact value: $\tan(15°)$

$\frac{1-\cos 30}{\sin 30} = \frac{1-\sqrt{3}/2}{1/2} \to 2-\sqrt{3}$

2. If $\sin u = \frac{4}{5}$ **in Q1, find** $\tan\frac{u}{2}$

$\cos u = \frac{3}{5} \to \frac{1-3/5}{4/5} = \frac{2/5}{4/5} \to \frac{1}{2}$

3. Simplify: $\frac{\sin 2x}{1+\cos 2x}$

This is the half-angle form for $\tan(\frac{2x}{2}) \to \tan x$