Solving Oblique Triangles

Precalculus Honors

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Unit Overview

We will develop and apply four powerful tools for solving any triangle — right or oblique.

- **Heron's Formula** — Find the area of a triangle using only its three side lengths
- **Law of Sines** — Relate sides and their opposite angles in any triangle
- **The Ambiguous Case (SSA)** — Determine how many triangles are possible given two sides and a non-included angle
- **Law of Cosines** — Generalize the Pythagorean Theorem to all triangles

Heron's Formula

Heron's Formula

When we know all three side lengths (a, b, c) but **not the height**, we can still find the area using the **semi-perimeter**.

$$s = \frac{a+b+c}{2} \qquad \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Why it works: Heron's Formula is derived by combining the standard area formula $A = \frac{1}{2}bh$ with the Law of Cosines to eliminate the height entirely.

*Named after **Hero of Alexandria** (~60 AD), though likely known to Archimedes centuries earlier.*

Heron's Formula — Worked Example

A triangular plot of land has sides of **40 m**, **50 m**, and **70 m**. Find its area.

$s = \frac{40+50+70}{2} = \frac{160}{2} = 80$

Area $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{80(80-40)(80-50)(80-70)}$

$= \sqrt{80 \cdot 40 \cdot 30 \cdot 10}$

$= \sqrt{960{,}000} \approx 979.8$ m^2

Interactive: Heron's Formula Explorer

```
//| panel: sidebar
viewof h_a = Inputs.range([1, 15], {value: 6, step: 0.5, la
viewof h_b = Inputs.range([1, 15], {value: 8, step: 0.5, la
viewof h_c = Inputs.range([1, 15], {value: 10, step: 0.5, l
```

```
//| panel: fill
h_valid = (h_a + h_b > h_c) && (h_a + h_c > h_b) && (h_b +
h_s = (h_a + h_b + h_c) / 2
h_area = h_valid ? Math.sqrt(h_s * (h_s - h_a) * (h_s - h_b
h_angleA = h_valid ? Math.acos((h_b**2 + h_c**2 - h_a**2) /

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-1, 16]},
  y: {domain: [-3, 10]},
  grid: true,
  marks: [
    h_valid ? Plot.line([
      [0, 0]
```

Heron's Formula — Problem 1

A triangular sail has sides of 9 ft, 12 ft, and 15 ft. Find its area.

$s = \frac{9+12+15}{2} = 18$

Area $= \sqrt{18(18-9)(18-12)(18-15)}$

$= \sqrt{18 \cdot 9 \cdot 6 \cdot 3} = \sqrt{2916}$

$= 54$ ft^2

Note: This is a 3-4-5 right triangle scaled by 3, so we can verify: $\frac{1}{2}(9)(12) = 54$

Heron's Formula — Problem 2

A triangular park has sides of 120 m, 170 m, and 250 m. Find its area.

$s = \frac{120+170+250}{2} = 270$

Area $= \sqrt{270(270-120)(270-170)(270-250)}$

$= \sqrt{270 \cdot 150 \cdot 100 \cdot 20}$

$= \sqrt{81{,}000{,}000}$

≈ 9000 m^2

Heron's Formula — Problem 3

Find the area of a triangle with sides $a = 7$, $b = 8$, $c = 5$.

$s = \frac{7+8+5}{2} = 10$

Area $= \sqrt{10(10-7)(10-8)(10-5)}$

$= \sqrt{10 \cdot 3 \cdot 2 \cdot 5} = \sqrt{300}$

$= 10\sqrt{3} \approx 17.32$ sq units

Heron's Formula — Problem 4

An equilateral triangle has a side length of 10 cm. Find its area using Heron's Formula.

$s = \frac{10+10+10}{2} = 15$

Area $= \sqrt{15(15-10)(15-10)(15-10)}$

$= \sqrt{15 \cdot 5 \cdot 5 \cdot 5} = \sqrt{1875}$

$= 25\sqrt{3} \approx 43.30$ cm^2

Verify with the standard formula: $A = \frac{\sqrt{3}}{4}(10)^2 = 25\sqrt{3}$

Heron's Formula — Problem 5

A triangular lot has sides of 35 yd, 65 yd, and 80 yd. A landscaper charges $0.15 per square yard to seed the lot. What is the total cost?

$s = \frac{35+65+80}{2} = 90$

Area $= \sqrt{90(90-35)(90-65)(90-80)}$

$= \sqrt{90 \cdot 55 \cdot 25 \cdot 10} = \sqrt{1{,}237{,}500}$

≈ 1112.5 yd^2

Cost $= 1112.5 \times \$0.15 \approx \$166.87$

Law of Sines

Law of Sines

In **any** triangle ABC with sides a, b, c opposite to angles A, B, C:

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Use the Law of Sines when you know:

Given	Find
AAS — two angles and a non-included side	sides
ASA — two angles and the included side	sides

Proof Sketch: *Draw altitude h from C. Then $h = b\sin A = a\sin B$, so $\frac{\sin A}{a} = \frac{\sin B}{b}$.*

Law of Sines — Worked Example

In $\triangle ABC$, $A = 40°$, $B = 75°$, and $a = 12$. Find side b.

$C = 180° - 40° - 75° = 65°$

Apply the Law of Sines:

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin 40°}{12} = \frac{\sin 75°}{b}$

$b = \frac{12 \sin 75°}{\sin 40°}$

$b \approx \frac{12(0.9659)}{0.6428} \approx 18.03$

Interactive: Law of Sines Explorer

```
//| panel: sidebar
viewof los_A = Inputs.range([10, 140], {value: 40, step: 1,
viewof los_B = Inputs.range([10, 140], {value: 75, step: 1,
viewof los_a = Inputs.range([1, 20], {value: 12, step: 0.5,
```

```
//| panel: fill
los_C_deg = 180 - los_A - los_B
los_valid = los_C_deg > 0
los_C = los_C_deg * Math.PI / 180
los_Ar = los_A * Math.PI / 180
los_Br = los_B * Math.PI / 180

los_b = los_valid ? (los_a * Math.sin(los_Br)) / Math.sin(l
los_c = los_valid ? (los_a * Math.sin(los_C)) / Math.sin(lo

// Place vertex A at origin, B at (c, 0), solve for C
los_Cx = los_valid ? los_b * Math.cos(los_Ar) : 0
los_Cy = los_valid ? los_b * Math.sin(los_Ar) : 0
```

Law of Sines — Problem 1

In $\triangle ABC$, $A = 35°$, $B = 85°$, and $c = 14$. Find side a.

$$C = 180° - 35° - 85° = 60°$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \implies \frac{\sin 35°}{a} = \frac{\sin 60°}{14}$$

$$a = \frac{14 \sin 35°}{\sin 60°}$$

$$a \approx \frac{14(0.5736)}{0.8660} \approx 9.27$$

Law of Sines — Problem 2

Two forest rangers at stations A and B, which are 10 miles apart, both spot a fire at point F. Ranger A measures angle $FAB = 52°$ and Ranger B measures angle $FBA = 61°$. How far is the fire from Station A?

$F = 180° - 52° - 61° = 67°$

$\frac{\sin F}{AB} = \frac{\sin B}{AF} \implies \frac{\sin 67°}{10} = \frac{\sin 61°}{AF}$

$AF = \frac{10 \sin 61°}{\sin 67°}$

$AF \approx \frac{10(0.8746)}{0.9205} \approx 9.50$ miles

Law of Sines — Problem 3

In $\triangle DEF$, $D = 110°$, $E = 30°$, and $d = 25$. Find side e.

$F = 180° - 110° - 30° = 40°$

$\frac{\sin D}{d} = \frac{\sin E}{e} \implies \frac{\sin 110°}{25} = \frac{\sin 30°}{e}$

$e = \frac{25 \sin 30°}{\sin 110°}$

$e \approx \frac{25(0.5)}{0.9397} \approx 13.30$

Law of Sines — Problem 4

A telephone pole leans 8° from vertical. From a point on the ground 40 ft from the base, the angle of elevation to the top of the pole is 32°. Find the length of the pole.

The angle at the base of the pole (from the ground) is $90° - 8° = 82°$.

Third angle $= 180° - 32° - 82° = 66°$

$\frac{\sin 66°}{\text{pole}} = \frac{\sin 32°}{40}$

pole $= \frac{40 \sin 66°}{\sin 32°}$

$\approx \frac{40(0.9135)}{0.5299} \approx 68.97$ ft

Law of Sines — Problem 5

In $\triangle PQR$, $P = 28°$, $Q = 112°$, and $p = 9.4$. Find side r and the area of the triangle.

$R = 180° - 28° - 112° = 40°$

$\frac{\sin 28°}{9.4} = \frac{\sin 40°}{r} \implies r = \frac{9.4 \sin 40°}{\sin 28°} \approx 12.87$

Find q : $\quad q = \frac{9.4 \sin 112°}{\sin 28°} \approx 18.58$

Area $= \frac{1}{2} \cdot p \cdot r \cdot \sin Q = \frac{1}{2}(9.4)(12.87) \sin 112° \approx 56.16$ sq units

The Ambiguous Case (SSA)

The Ambiguous Case — Overview

When given **SSA** (two sides and an angle *opposite* one of them), there may be **0, 1, or 2** valid triangles.

Given sides a and b and angle A:

Condition	Number of Triangles
$a < b \sin A$	**0** — side a too short to reach
$a = b \sin A$	**1** — right triangle
$b \sin A < a < b$	**2** — two triangles possible
$a \geq b$	**1** — side a long enough

The ambiguity arises because $\sin\theta = \sin(180° - \theta)$, so there may be ***two angles*** *with the same sine value.*

Ambiguous Case — How to Find Both Solutions

Step 1: Use $\frac{\sin B}{b} = \frac{\sin A}{a}$ to find $\sin B$.

Step 2: Find $B_1 = \arcsin(\sin B)$ (acute solution).

Step 3: Check $B_2 = 180° - B_1$ (obtuse solution). It is valid **only if** $A + B_2 < 180°$.

Triangle 1: $A, B_1, C_1 = 180° - A - B_1$

Triangle 2: $A, B_2, C_2 = 180° - A - B_2$

Then use the Law of Sines to solve each triangle completely.

Interactive: Ambiguous Case Visualizer

```
//| panel: sidebar
viewof ssa_A = Inputs.range([5, 85], {value: 35, step: 1, l
viewof ssa_b = Inputs.range([1, 20], {value: 10, step: 0.5,
viewof ssa_a = Inputs.range([1, 20], {value: 6, step: 0.5,
```

```
//| panel: fill
ssa_Ar = ssa_A * Math.PI / 180
ssa_h = ssa_b * Math.sin(ssa_Ar)
ssa_sinB = (ssa_b * Math.sin(ssa_Ar)) / ssa_a
ssa_B1r = ssa_sinB <= 1 ? Math.asin(ssa_sinB) : null
ssa_B1 = ssa_B1r ? ssa_B1r * 180 / Math.PI : null
ssa_B2 = ssa_B1 ? 180 - ssa_B1 : null
ssa_valid2 = ssa_B2 && (ssa_A + ssa_B2 < 180)

// Triangle 1 geometry
ssa_C1 = ssa_B1 ? (180 - ssa_A - ssa_B1) * Math.PI / 180 :
ssa_c1 = ssa_C1 ? (ssa_a * Math.sin(ssa_C1)) / Math.sin(ssa

// Triangle 2 geometry
```

Ambiguous Case — Problem 1

Given $A = 30°$**,** $a = 6$**,** $b = 10$**. How many triangles exist? Solve.**

$h = b \sin A = 10 \sin 30° = 5$

Since $h = 5 < a = 6 < b = 10$: **Two triangles possible.**

$\sin B = \frac{b \sin A}{a} = \frac{10(0.5)}{6} \approx 0.8333$

$B_1 \approx 56.4°, \quad C_1 \approx 93.6°, \quad c_1 = \frac{6 \sin 93.6°}{\sin 30°} \approx 11.95$

$B_2 \approx 123.6°, \quad C_2 \approx 26.4°, \quad c_2 = \frac{6 \sin 26.4°}{\sin 30°} \approx 5.35$

Answer: Two triangles.

Ambiguous Case — Problem 2

Given $A = 50°$, $a = 4$, $b = 6$. How many triangles? Solve completely.

$h = 6 \sin 50° \approx 4.60$

Since $a = 4 < h \approx 4.60$: **No triangle is possible.**

$\sin B = \frac{6 \sin 50°}{4} \approx 1.149 > 1$

Answer: No solution.

The side opposite the given angle is too short to reach the base.

Ambiguous Case — Problem 3

Given $A = 45°$, $a = 7\sqrt{2}$, $b = 14$. How many triangles? Solve.

$h = 14 \sin 45° = 14 \cdot \frac{\sqrt{2}}{2} = 7\sqrt{2}$

Since $a = 7\sqrt{2} = h$: **Exactly one right triangle.**

$\sin B = \frac{14 \sin 45°}{7\sqrt{2}} = 1 \implies B = 90°$

$C = 180° - 45° - 90° = 45°$

$c = \frac{7\sqrt{2} \cdot \sin 45°}{\sin 45°} = 7\sqrt{2} \approx 9.90$

Ambiguous Case — Problem 4

Given $A = 60°$**,** $a = 15$**,** $b = 10$**. How many triangles? Solve.**

$h = 10 \sin 60° = 10 \cdot \frac{\sqrt{3}}{2} \approx 8.66$

Since $a = 15 > b = 10$: **Exactly one triangle.**

$\sin B = \frac{10 \sin 60°}{15} = \frac{10 \cdot 0.8660}{15} \approx 0.5774 \implies B \approx 35.3°$

(The obtuse case $B_2 \approx 144.7°$ *fails since* $60° + 144.7° > 180°$*)*

$C \approx 180° - 60° - 35.3° = 84.7°$

$c = \frac{15 \sin 84.7°}{\sin 60°} \approx 17.22$

Ambiguous Case — Problem 5

A surveyor needs to locate a boundary marker. From point A, she knows angle $A = 38°$, side $b = 20$ m, and side $a = 14$ m. How many positions are possible for the third vertex?

$h = 20 \sin 38° \approx 12.31$

Since $h \approx 12.31 < a = 14 < b = 20$: **Two positions are possible.**

$\sin B = \frac{20 \sin 38°}{14} \approx 0.8791$

$B_1 \approx 61.5°, \quad C_1 \approx 80.5°, \quad c_1 \approx \frac{14 \sin 80.5°}{\sin 38°} \approx 22.41$ m

$B_2 \approx 118.5°, \quad C_2 \approx 23.5°, \quad c_2 \approx \frac{14 \sin 23.5°}{\sin 38°} \approx 9.02$ m

Answer: Two positions are possible.

Law of Cosines

Law of Cosines

A generalization of the Pythagorean Theorem to **any** triangle:

$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Solving for an angle:

$\cos A = \frac{b^2+c^2-a^2}{2bc}$

Use when you know:

Given	Find
SAS — two sides and the **included** angle	third side
SSS — all three sides	any angle

Law of Cosines — Worked Example

Two ships leave port. Ship A travels 50 mi at $N\,40^\circ\,E$ and Ship B travels 35 mi at $S\,50^\circ\,E$. How far apart are the ships?

The angle between them is $40^\circ + 50^\circ = 90^\circ$… but let's say the angle is actually 130° for a richer example.

$d^2 = 50^2 + 35^2 - 2(50)(35)\cos(130^\circ)$

$= 2500 + 1225 - 3500(-0.6428)$

$= 3725 + 2249.8 = 5974.8$

$d = \sqrt{5974.8} \approx 77.3$ miles

Interactive: Law of Cosines Explorer

```
//| panel: sidebar
viewof loc_b = Inputs.range([1, 20], {value: 10, step: 0.5,
viewof loc_c = Inputs.range([1, 20], {value: 12, step: 0.5,
viewof loc_A_deg = Inputs.range([5, 175], {value: 60, step:
```

```
//| panel: fill
loc_Ar = loc_A_deg * Math.PI / 180
loc_a_sq = loc_b**2 + loc_c**2 - 2*loc_b*loc_c*Math.cos(loc
loc_a = Math.sqrt(loc_a_sq)

loc_cosB = (loc_a_sq + loc_c**2 - loc_b**2) / (2 * loc_a *
loc_cosC = (loc_a_sq + loc_b**2 - loc_c**2) / (2 * loc_a *
loc_B = Math.acos(loc_cosB) * 180 / Math.PI
loc_C = Math.acos(loc_cosC) * 180 / Math.PI

// Draw: A at origin, C at (b, 0), B somewhere
loc_Bx = loc_c
loc_By = 0
loc_Cx = loc_b * Math.cos(loc_Ar)
```

Law of Cosines — Problem 1

In $\triangle ABC$, $a = 8$, $b = 11$, $C = 42°$. Find side c.

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 8^2 + 11^2 - 2(8)(11)\cos 42°$

$c^2 = 64 + 121 - 176(0.7431)$

$c^2 = 185 - 130.79 = 54.21$

$c = \sqrt{54.21} \approx 7.36$

Law of Cosines — Problem 2

A triangular plot has sides $a = 25$ m, $b = 32$ m, $c = 40$ m. Find the largest angle.

The largest angle is opposite the longest side $c = 40$.

$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{25^2+32^2-40^2}{2(25)(32)}$

$\cos C = \frac{625+1024-1600}{1600} = \frac{49}{1600} \approx 0.0306$

$C = \arccos(0.0306) \approx 88.2°$

Law of Cosines — Problem 3

Two tugboats pull a barge. Their cables make an angle of $50°$ with each other. The cable lengths are 120 ft and 150 ft. How far apart are the tugboats?

$d^2 = 120^2 + 150^2 - 2(120)(150)\cos 50°$

$= 14400 + 22500 - 36000(0.6428)$

$= 36900 - 23140.8 = 13759.2$

$d = \sqrt{13759.2} \approx 117.3$ ft

Law of Cosines — Problem 4

A parallelogram has sides of 12 cm and 17 cm. The angle between them is $65°$. Find the length of the longer diagonal.

The longer diagonal is opposite the **obtuse** interior angle: $180° - 65° = 115°$.

$d^2 = 12^2 + 17^2 - 2(12)(17)\cos(115°)$

$= 144 + 289 - 408(-0.4226)$

$= 433 + 172.42 = 605.42$

$d = \sqrt{605.42} \approx 24.61$ cm

Law of Cosines — Problem 5

A golfer hits a ball 200 yards due north. A strong wind pushes it so its final position is 180 yards from the tee at a bearing of $N\,25^\circ\,E$. How far did the ball drift east from the intended landing spot?

The angle between the 200-yd path and the 180-yd actual distance is 25°.

$d^2 = 200^2 + 180^2 - 2(200)(180)\cos 25^\circ$

$= 40000 + 32400 - 72000(0.9063)$

$= 72400 - 65253.6 = 7146.4$

$d = \sqrt{7146.4} \approx 84.5$ yards

Choosing the Right Law

Use this decision tree to choose your approach for any oblique triangle.

```
html`<div style="font-family: Georgia, serif; font-size: 14
<p><strong style="color:#2c3e50">What information do you ha
<ul>
  <li><strong style="color:#2980b9">AAS or ASA</strong> → <
  <li><strong style="color:#e67e22">SSA</strong> → <span st
  <li><strong style="color:#8e44ad">SAS</strong> → <span st
  <li><strong style="color:#c0392b">SSS</strong> → <span st
  <li><strong style="color:#16a085">SSS (area only)</strong
</ul>
<p><em>Always verify: angles sum to 180°, triangle inequali
</div>`
```

Summary

Tool	Given	Finds
Heron's Formula	a, b, c	Area
Law of Sines	AAS, ASA	Remaining sides/angles
Ambiguous Case	SSA	0, 1, or 2 triangles
Law of Cosines	SAS or SSS	Third side or any angle

$s = \frac{a+b+c}{2}, \quad \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$c^2 = a^2 + b^2 - 2ab\cos C \quad \Leftrightarrow \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$