Unit 10: Matrices

Precalculus Honors | Standards PC.NR.1 & PC.PAFR.7

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Standards Overview

We will explore the representation and manipulation of data using matrices and using them in solving complex systems.

- **Matrix Operations (PC.NR.1)** — Addition, subtraction, and multiplication. Exploring identity/zero matrices and the non-commutative nature of multiplication.
- **Inverses & Determinants (PC.NR.1)** — Finding additive and multiplicative inverses and using determinants to verify inverse existence.
- **Systems & Modeling (PC.PAFR.7)** — Solving linear-quadratic systems and representing linear systems as matrix equations ($AX = B$).

Matrix Operations & Properties

Basic Definitions

Identity Matrix (I): A square matrix with 1s on the main diagonal and 0s elsewhere. Multiplying any matrix A by I results in A.

Zero Matrix (0): A matrix of any dimension where every element is 0. Adding it to A results in A.

Commutativity: Unlike real numbers, matrix multiplication is **not commutative**. In general, $AB \neq BA$.

Interactive: Matrix Commutativity Check

```
//| panel: sidebar
// Matrix A Inputs
viewof a11 = Inputs.number({value: 1, label: "A[1,1]"})
viewof a12 = Inputs.number({value: 2, label: "A[1,2]"})
viewof a21 = Inputs.number({value: 3, label: "A[2,1]"})
viewof a22 = Inputs.number({value: 4, label: "A[2,2]"})

// Matrix B Inputs
viewof b11 = Inputs.number({value: 5, label: "B[1,1]"})
viewof b12 = Inputs.number({value: 6, label: "B[1,2]"})
viewof b21 = Inputs.number({value: 7, label: "B[2,1]"})
viewof b22 = Inputs.number({value: 8, label: "B[2,2]"})

// Show/Hide Toggle
viewof showProducts = Inputs.toggle({label: "Show Product
```

```
//| panel: fill
// Calculations for AB
ab11 = a11*b11 + a12*b21; ab12 = a11*b12 + a12*b22;
```

Practice: PC.NR.1.1

1. Find $A+B$ **if** $A=\begin{bmatrix}2 & -1\\4 & 0\end{bmatrix}$ **and** $B=\begin{bmatrix}5 & 3\\-2 & 1\end{bmatrix}$**.**

$$\begin{bmatrix}2+5 & -1+3\\4-2 & 0+1\end{bmatrix}\rightarrow\begin{bmatrix}7 & 2\\2 & 1\end{bmatrix}$$

2. Multiply $C=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ **by scalar** $k=-3$**.**

$$\begin{bmatrix}-3(1) & -3(2)\\-3(3) & -3(4)\end{bmatrix}\rightarrow\begin{bmatrix}-3 & -6\\-9 & -12\end{bmatrix}$$

3. Find the product AB if $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$.

$$\begin{bmatrix} (1)(3) + (0)(4) & (1)(1) + (0)(2) \\ (2)(3) + (1)(4) & (2)(1) + (1)(2) \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 \\ 10 & 4 \end{bmatrix}$$

4. True or False: If A is 2×3 and B is 3×2, then AB and BA are both defined.

True. AB is 2×2 and BA is 3×3. However, they cannot be equal because their dimensions differ!

5. Identify the 3×3 Identity Matrix I_3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinants & Inverses

The Role of the Determinant

For a square matrix A, the **determinant** ($\det A$ or $|A|$) is a scalar value that determines if an inverse exists.

Invertibility: A square matrix A has a multiplicative inverse A^{-1} **if and only if** $\det A \neq 0$.

2x2 Formula: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$.

Multiplicative Inverse: $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Practice: PC.NR.1.2 - 1.4

1. Find the determinant of $A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$**.**

$(3)(4) - (2)(5) = 12 - 10 \rightarrow$ 2

2. Does the matrix $B = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ **have an inverse?**

$\det B = (6)(2) - (3)(4) = 12 - 12 = 0 \rightarrow$ No inverse exists (singular matrix).

3. Find the additive inverse of $C = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}$.

$-C = \begin{bmatrix} 1 & -5 \\ 0 & 2 \end{bmatrix}$

4. Calculate A^{-1} for $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$.

$\det A = 8 - 7 = 1$. $A^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$

5. If $\det(M) = 5$, what is $\det(M^{-1})$?

$\det(M^{-1}) = \frac{1}{\det M} \to \frac{1}{5}$

Systems of Equations

Linear-Quadratic Systems

A system consisting of a line and a parabola can have **0, 1, or 2** real solutions.

Algebraic Path: Set the equations equal $(f(x) = g(x))$ and solve the resulting quadratic equation.

Graphical Path: Identify the points of intersection on the coordinate plane.

Matrix Equations: $AX = B$

Any system of linear equations can be written as a single matrix equation:

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \implies \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- A**:** Coefficient Matrix
- X**:** Variable Vector
- B**:** Output/Constant Vector

To solve: $X = A^{-1}B$

Practice: PC.PAFR.7

1. Solve the system: $y = x^2$ **and** $y = x + 2$**.**

$x^2 = x + 2 \to x^2 - x - 2 = 0 \to (x-2)(x+1) = 0$ Solutions: $(2, 4)$ and $(-1, 1)$

2. How many solutions does $y = x^2 + 5$ **and** $y = 2$ **have?**

$x^2 + 5 = 2 \to x^2 = -3 \to$ 0 real solutions (no intersection).

3. Represent as $AX = B$**:** $\begin{cases} 3x - 2y = 10 \\ 5x + y = 7 \end{cases}$

$$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

4. Use a graph to solve $\sqrt{x+3} = x+1$.

Points of intersection for $f(x) = \sqrt{x+3}$ and $g(x) = x+1$.
Intersection at $x = 1$.

5. Solve $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ using $X = A^{-1}B$.

$$\det A = -2, A^{-1} = -\tfrac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.5(6) + 0.5(2) \\ 0.5(6) - 0.5(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Summary

Feature	Key Concept	Constraint
Multiplication	$AB \neq BA$	Inner dimensions must match
Inverse	$A \cdot A^{-1} = I$	$\det A \neq 0$
Systems	$AX = B$	$X = A^{-1}B$
Lin-Quad	Intersection points	Max 2 solutions

$$ad - bc = \det A \qquad A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad f(x) = g(x)$$

Geometric Interpretations (PC.NR.3.2)

- **Addition:** Follows the parallelogram rule (identical to vectors).
- **Conjugation:** A reflection over the Real (x) axis.
- **Multiplication by i:** A 90° counter-clockwise rotation.
- **Multiplication by $r(\cos\theta + i\sin\theta)$:** A scaling by r and rotation by θ.